J Glob Optim (2009) 45:131–151
DOI 10.1007/s10898-008-9398-3

Global optimization of multi-parametric MILP problems

N. P. Faísca · V. D. Kosmidis · B. Rustem · E. N. Pistikopoulos

Received: 11 December 2008 / Accepted: 18 December 2008 / Published online: 10 February 2009

Abstract In this paper, we present a novel global optimisation approach for the general solution of multi-parametric mixed integer linear programs (mp-MILPs). We describe an optimisation procedure which iterates between a (master) mixed integer nonlinear program and a (slave) multi-parametric program. Moreover, we explain how to overcome the presence of bilinearities, responsible for the non-convexity of the multi-parametric program, in two classes of mp-MILPs, with (i) varying parameters in the objective function and (ii) simultaneous presence of varying parameters in the objective function and the right-hand side of the constraints. Examples are provided to illustrate the solution steps.

Keywords Multi-parametric mixed-integer linear programming · Global optimization

1 Introduction

Multi-parametric programming refers to a class of optimisation problems which involve some type of bounded uncertainty and/or variability within the mathematical model. A typical and general multi-parametric program is of the following form:

$$\begin{aligned} z(\theta) = \min_{x,y} \quad & f(x, y, \theta), \\ s.t. \quad & h(x, y, \theta) = 0, \\ & g(x, y, \theta) \leq 0, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^m, \\ & \theta \in \Theta, \end{aligned} \tag{1}$$

N. P. Faísca · B. Rustem · E. N. Pistikopoulos (✉)
Centre for Process Systems Engineering, Imperial College London, London SW7 2AZ, UK
e-mail: e.pistikopoulos@imperial.ac.uk

V. D. Kosmidis
EDF Trading Markets Limited, 71 High Holborn, London WC1V 6ED, UK

Table 1 Classes of multi-parametric programming algorithms

mp-LP	[10,14,16,29,31–33,52]
mp-QP	[11,20,62]
mp-MILP	[1,18,34,37,38]
mp-MIQP	[20]
mp-NLP	[2,5,6,17,28,36]
mp-MINLP	[17,43,44]
mp-GO	[21]
mp-DO	[54–57]

Table 2 Applications of multi-parametric programming to MPC

Hierarchical decentralised control	[23]
Hybrid control	[8,15,42,54]
Linear discrete systems	[11,20,45,59,62]
Non-linear control	[60]
Robust control	[12,35,57,58]

Table 3 Other applications of multi-parametric programming

Drug delivery systems	[19,22]
Dynamic programming	[8,25]
Game theory	[23,24]
Pro-active scheduling	[50,51]
Supply chain	[48]

where, x is the vector of continuous optimisation variables, y is the vector of binary optimisation variables and θ is the vector of bounded parameters ($\theta^L \le \theta \le \theta^U$); f is a scalar function and h, g are general vectorial functions.

Multi-parametric programming has recently received considerable attention in the open literature (see [46]), especially due to its important application to model predictive control (MPC)—see [47]. Various classes of (1) have been studied, see Table 1, with corresponding important developments and applications in control, see Table 2, and other areas, Table 3.

Despite the above major advances, many important classes of Problem (1) have not yet been fully addressed. One important class is the general multi-parametric mixed integer linear program, of the following form:

$$
\begin{aligned}
z(\theta) = \min_{x,y} \quad & c(\theta)^T \cdot x + d(\theta)^T \cdot y, \\
s.t. \quad & A(\theta)x + E(\theta)y \le b(\theta), \\
& \Gamma(\theta)x + \Phi(\theta)y = \gamma(\theta), \\
& x \in X \subseteq \mathbb{R}^n, \quad y \in \{0,1\}^m, \\
& \theta \in \Theta.
\end{aligned}
\tag{2}
$$

where, c, d, A, E, Γ, Φ, b and γ are real matrices linearly dependent on θ, with appropriate dimensions. Problem (2) has applications in many areas, including pro-active scheduling and hybrid control. Logical decisions or switches are described by binary variables, as for instance in hybrid control the description of non-linear systems by sets of piecewise convex approximations.

Acevedo and Pistikopoulos [1] proposed the first multiparametric algorithm to address mixed-integer linear programming problems, involving varying parameters in the right-hand side of the constraints. The algorithm which was based on a branch and bound approach, solving at each node a mp-LP problem, was shown to be computationally expensive. Pertsinidis et al. [44] proposed an algorithm based on a master/slave iteration procedure, for the solution of single parameter problems (i.e. θ is scalar); while Dua and Pistikopoulos [18] extended their approach to address the general multi-parametric case. Recently, Li and Ierapetritou [37,38] presented an algorithm to address general multi-parametric linear programs, based on a branch and bound approach.

In this work, we first describe a novel algorithm to solve multi-parametric mixed-integer linear programming problems, involving varying parameters in the objective function (OFC mp-MILP):

$$\begin{aligned} z(\theta) = \min_{x,y} \quad & c(\theta)^T \cdot x + d(\theta)^T \cdot y, \\ s.t. \quad & Ax + Ey \leq b, \\ & \Gamma x + \Phi y = \gamma, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0,1\}^m, \\ & \theta \in \Theta, \end{aligned} \tag{3}$$

and then, an extension for the solution of general multi-parametric mixed-integer linear programming problems, involving varying parameters in the objective function and the right-hand side of the constraints (RIM mp-MILP):

$$\begin{aligned} z(\theta) = \min_{x,y} \quad & c(\theta)^T \cdot x + d(\theta)^T \cdot y, \\ s.t. \quad & Ax + Ey \leq b(\theta), \\ & \Gamma x + \Phi y = \gamma(\theta), \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0,1\}^m, \\ & \theta \in \Theta. \end{aligned} \tag{4}$$

The algorithms described hereto follow the early developments presented in Pertsinidis et al. [44] and Dua and Pistikopoulos [18]. The principal idea is to iterate between a master problem, where we solve to global optimality a mixed integer non-linear program (MINLP); and a slave problem, involving the solution of a multi-parametric program, obtained by fixing the binary variables to the previously computed optimal MINLP solution. The main challenge resides in the slave problem, because the presence of bilinearities implies that it is a non-convex problem. Notwithstanding, we circumvent the use of global optimisation tools. We develop a new multi-parametric linear programming (mp-LP) algorithm, based on the sensitivity theory [27] and singularity theory [7], which easily handles the bilinearities and frees the slave problem from the need of any global optimisation procedure [21].

This paper is organised in the following way. In Sect. 2 we describe the two-stage approach to solve OFC mp-MILP problems as in (3). Then, we present the master problem and its solution steps, and then, we present a detailed description of the new mp-LP algorithm, establishing the links with our previous work [46]. Section 3 describes a general procedure to address RIM mp-MILP problems as in (4). Illustrative examples are presented throughout the sections to provide details of the steps of the proposed algorithms.

2 Multi-parametric OFC MILP problems

Consider the formulation in (3), rewritten in a more compact mathematical form [34]:

$$\begin{aligned} z(\theta) = \min_{x,y} \quad & (c + H\theta)^T x + (d + L\theta)^T y, \\ s.t. \quad & Ax + Ey \leq b, \\ & \Gamma x + \Phi y = \gamma, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^m, \\ & \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}, \end{aligned} \tag{5}$$

here, c, d, H and L are real matrices with appropriate dimensions. The presence of parametric uncertainties in the objective function introduces two types of bilinear terms—$\theta^T \cdot H^T \cdot x$ and $\theta^T \cdot H^T \cdot y$—hence, this is a non-convex objective function. Here, we propose a two-stage global optimisation procedure for the solution of (5), described next.

2.1 Master MINLP problem

In the master problem, we formulate a global optimisation problem considering the varying parameters, θ, as bounded optimisation variables:

$$\begin{aligned} z_M(\theta) = \min_{x,y,\theta} \quad & (c + H\theta)^T x + (d + L\theta)^T y, \\ s.t. \quad & Ax + Ey \leq b, \\ & \Gamma x + \Phi y = \gamma, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^m, \\ & \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}. \end{aligned} \tag{6}$$

Problem (6) is solved using a global optimisation solver [3,4,30,61]. In this work, we use the commercial package GAMS/BARON [53] as our global optimisation solver. From the solution obtained for Problem (6), the binary vector is fixed, $y = \bar{y}$, and is an entry data in the slave problem, which is described next.

2.2 Slave mp-LP problem

Fixing $y = \bar{y}$, (5) results in the following formulation:

$$\begin{aligned} z_S(\theta) = (d + L\theta)^T \bar{y} + \min_{x} \quad & c^T x + \theta^T H^T x, \\ s.t. \quad & Ax \leq b', \\ & \Gamma x = \gamma', \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^m, \\ & \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}, \end{aligned} \tag{7}$$

where, $b' = b - E\bar{y}$, and $\gamma' = \gamma - \Phi\bar{y}$. Problem (7) involves bilinear terms, and hence, it corresponds to a multi-parametric global optimisation problem.

In principle, (7) can be addressed by applying the global optimisation algorithm of Dua et al. [21]. However, here we explore the structure of (7) to design a new mp-LP algorithm suitable for OFC mp-LP problems.

2.2.1 Algorithm for the OFC mp-LP problem

The Fritz John first-order conditions state that there exist $p + q + 1$ real numbers ν, λ, μ, not all zero, such that [40]:

$$\mathscr{L}(x, \nu, \lambda, \mu, \theta) = \nu f(x, \theta) + \sum_{i=1}^{p} \lambda_i g_i(x, \theta) + \sum_{j=1}^{q} \mu_j h_j(x, \theta),$$

$$\begin{aligned} \nabla_x \mathscr{L}(x, \nu, \lambda, \mu, \theta) = 0, \\ \lambda_i g_i(x, \theta) = 0, \quad \forall i = 1, \ldots, p, \\ h_j(x, \theta) = 0, \quad \forall j = 1, \ldots, q, \\ \nu, \lambda_i, \mu_j \geq 0, \end{aligned} \tag{8}$$

where, $\mathscr{L}(x, \nu, \lambda, \mu)$ is the Lagrangian, $\nu \in \mathbb{R}, \lambda \in \mathbb{R}^p, \mu \in \mathbb{R}^q$, are the Lagrange multipliers, $f(x, \theta) = c^T x + \theta^T H^T x$, $g(x, \theta) = Ax - b' \leq 0$ and $h(x, \theta) = \Gamma x - \gamma' = 0$. Assuming we seek a Karush–Kuhn–Tucker (KKT) optimum [9], $\nu = 1$, (8) is rewritten in a more compact form:

$$F(\eta, \theta) = \begin{bmatrix} \nabla_x \mathscr{L} \\ \Lambda g(x, \theta) \\ h(x, \theta) \end{bmatrix} = 0, \tag{9}$$

here, $\eta = [x, \lambda, \mu]$ and Λ is a diagonal matrix with $\Lambda_{ii} = \lambda_i, i = 1, \ldots, p$. Then, deriving (9) with respect to η and to θ, and using the chain rule, we obtain an expression of the optimal solution of (7) as an explicit function of θ, as shown next in Theorem 1.

Theorem 1 *Basic Sensitivity Theorem [26]: Let θ_0 be a vector of parameter values and (x_0, λ_0, μ_0) a KKT triple corresponding to (8), where λ_0 is nonnegative and x_0 is feasible in (7). Also assume that (i) strict complementary slackness (SCS) holds, (ii) the binding constraint gradients are linearly independent (LICQ: Linear Independence Constraint Qualification), and (iii) the second-order sufficiency conditions (SOSC) hold. Then, in the neighbourhood of θ_0, there exists a unique, once continuously differentiable function, $\eta(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]$, satisfying (8) with $\eta(\theta_0) = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)]$, where $x(\theta)$ is a unique isolated minimiser of (7), and*

$$\begin{pmatrix} \frac{dx}{d\theta} \\ \frac{d\lambda}{d\theta} \\ \frac{d\mu}{d\theta} \end{pmatrix} = -(M_0)^{-1} N_0, \tag{10}$$

where, M_0 and N_0 are the Jacobians of (9) with respect to η and θ,

$$M_0 = \left(\begin{array}{c|ccc|ccc} \nabla^2_{xx}\mathscr{L} & \nabla_x g_1 & \cdots & \nabla_x g_p & \nabla_x h_1 & \cdots & \nabla_x h_q \\ \hline \lambda_1 \nabla_x^T g_1 & g_1 & & & & & \\ \vdots & & \ddots & & & 0 & \\ \lambda_p \nabla_x^T g_p & & & g_p & & & \\ \hline \nabla_x^T h_1 & & & & & & \\ \vdots & & 0 & & & 0 & \\ \nabla_x^T h_q & & & & & & \end{array} \right),$$

$$N_0 = (\nabla^2_{\theta x}\mathscr{L}, \lambda_1 \nabla_\theta^T g_1, \ldots, \lambda_p \nabla_\theta^T g_p, \nabla_\theta^T h_1, \ldots, \nabla_\theta^T h_q)^T.$$

Proof See [27, p. 72]. □

Note that the assumptions stated in the theorem above ensure M_0 is non-singular in the neighbourhood of the solution point (η_0, θ_0), and hence, invertible [41].

Corollary 1 *First-order estimation of* $[x(\theta), \lambda(\theta), \mu(\theta)]$*, near* $\theta = \theta_0$ *[27]: Under the assumptions of Theorem 1, a first-order approximation of* $[x(\theta), \lambda(\theta), \mu(\theta)]$ *in the neighbourhood of* θ_0 *is,*

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - (M_0)^{-1} \cdot N_0 \cdot (\theta - \theta_0) + o(||\theta||), \tag{11}$$

where $(x_0, \lambda_0, \mu_0) = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)]$, $M_0 = M(\theta_0)$, $N_0 = N(\theta_0)$, *and* $\phi(\theta) = o(||\theta||)$ *means that* $\phi(\theta)/||\theta|| \to 0$ *as* $\theta \to \theta_0$.

From Theorem 1, it is obvious that the matrices M_0 and N_0 are independent of θ, i.e. it is equally applicable to (7) as it is for the righ-hand-side (RHS) case considered in Dua and Pistikopoulos [18]. Theorem 1 clearly states that the first order estimation of the explicit optimal function, (11), is the general solution inside the incumbent critical region, where a critical region is defined as a subset of the parameters space inside which the same set of active constraints applies.

The main difference between the RHS case and the OFC mp-LP problem, in (7), is the non-null Hessian of the Lagrangian with respect to θ and x, i.e. $\nabla_{\theta x}\mathscr{L} = H^T$. Yet, all matrices in (11) are constant, and hence, the explicit expression is indeed valid inside the entire critical region. By substituting the appropriate variables, (11) results in the following expression for (7):

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - \begin{bmatrix} 0 & A^T & \Gamma^T \\ \Lambda A & diag\,(g) & 0 \\ \Gamma & 0 & 0 \end{bmatrix}_{\eta_0}^{-1} \cdot \begin{bmatrix} H^T \\ 0 \\ 0 \end{bmatrix}_{\eta_0} \cdot (\theta - \theta_0). \tag{12}$$

The analytical expressions in (12) are used to derive the boundaries of the critical region by checking the conditions stated in the following Theorem 2.

Theorem 2 [49] *Let* $(\eta_0, \nu_0, \theta_0)$ *be a solution to* (8)*. Additionally, assume that* f*,* g *and* h *are twice continuously differentiable in a neighbourhood of* (x_0, θ_0)*, and define two index sets:* $\mathcal{A}$ *and* $\bar{\mathcal{A}}$*, and a corresponding tangent space* $\bar{\mathcal{T}}$ *by,*

$$\bar{\mathcal{A}} = \{i : 1 \le i \le p, g_i(x_0, \theta_0) = 0\}, \tag{13a}$$

$$\mathcal{A} = \{i \in \bar{\mathcal{A}} : \lambda_i \neq 0\}, \tag{13b}$$

$$\bar{\mathcal{T}} = \{t \in \mathbb{R}^n : [\nabla_x h(x_0, \theta_0)]^T y = 0, [\nabla_x g_i(x_0, \theta_0)]^T y = 0, \forall i \in \bar{\mathcal{A}}\}. \tag{13c}$$

Then a necessary and sufficient condition for $\nabla_\eta F$*, i.e.* M_0*, to be non-singular is that, each of the following three conditions hold:*

(i) $\bar{\mathcal{A}} = \mathcal{A}$;
(ii) $S \triangleq \{\nabla_x g_i(x_0, \theta_0) \cup \nabla_x h_j(x_0, \theta_0), i \in \bar{\mathcal{A}}, j = 1, \ldots, q\}$ *is a linearly independent collection of* $q + |\bar{\mathcal{A}}|$ *vectors, where* $|\cdot|$ *denotes cardinality;*
(iii) *The Hessian of the Lagrangian* $\nabla_x \mathscr{L}$ *is non-singular on the tangent space* $\bar{\mathcal{T}}$.

If $\nabla_\eta F(\eta_0, \theta_0)$ *is non-singular, there exist neighbourhoods* $\mathcal{B}_1$ *of* θ_0 *and* $\mathcal{B}_2$ *of* (η_0, θ_0) *and a function* $\phi \in C^1(\mathcal{B}_1)$ *such that* $F(\phi(\theta), \theta) = 0, \forall_\theta \in \mathcal{B}_1$ *and* $\phi(\theta_0) = \eta_0$*. This solution is locally unique in the sense that if* $(\eta, \theta) \in \mathcal{B}_2$ *and* $F(\eta, \theta) = 0$*, then* η *belongs to the manifold defined by* ϕ*, i.e.,* $\eta = \phi(\theta)$*. Furthermore, if* f*,* g *and* h *are* $C^k (k \geq 2)$ $(C^\infty$ *or real analytic*) *then* ϕ *is* $C^{k-1}(C^\infty$ *or real analytic, respectively) on* $\mathcal{B}_1$*.*

Proof See Poore and Tiahrt [49]. □

Essentially, the singular point/surface occurs when at least one of the three conditions enumerated in Theorem 2 is violated: (i) loss of strict complementarity, which is identified by any change of sign or occurrence of zeros in any of the inactive constraints or active inequality Lagrange multiplier; (ii) violation of the linear independence constraint qualification, identified by a change of sign or the occurrence of a zero in ν; and (iii) singularity of the Hessian of the Lagrangian on the tangent space to the active constraints, which is identified by a change in in$(\nabla_x^2 \mathcal{L}_T)$, where the operator in $(\cdot)$ represents the inertia of the matrix. By inertia of a matrix we understand the number of positive, negative and zero eigenvalues [39].

However, since we have a KKT point computed in the master problem, (6), and an explicit optimal function, (12), the limits for the validity of these explicit expressions can be resumed by the following [20]:

$$\tilde{\lambda}(x(\theta)) \geq 0, \tag{14}$$

$$\breve{g}(x(\theta)) \leq 0, \tag{15}$$

where, $\tilde{\lambda}$ represents the set of Lagrange multipliers of the active constraints, $\mathcal{A}$, and $\breve{g}$ the set of inactive constraints.

The parameters' initial area is further explored using the methodology described by Dua and Pistikopoulos [18]—see Appendix. At the end, a complete map of all critical regions is obtained. Each critical region is associated with a corresponding analytical expression as in (12). By substituting this expression in z_S, (7), and being the second order sufficient conditions satisfied, a valid upper bound is obtained for (5).

The OFC mp-LP algorithm was implemented in Matlab.

Remark 1 Note that optimisation variables, x, in (12), are independent of θ, inside each critical region, $x \neq f(\theta)$; this is expected as (7) has uncertainty only in the objective function and it is linear with respect to x. Of course, the varying parameters, θ, continue to affect the optimal value function, z^*.

2.3 The algorithm

Between every master–slave iteration, we need to (i) introduce integer and parametric cuts in the master problem (MINLP), in order to avoid already examined 0–1 combinations and cut off worse solutions. For (i), we introduce the following constraints [18]:

$$\sum_{j \in J^{ik}} y_j^{ik} - \sum_{j \in L^{ik}} y_j^{ik} \leq |J^{ik}| - 1, \quad k = 1, \ldots, K^i, \tag{16}$$

$$(c + H\theta)^T x + (d + L\theta)^T y \leq z_S(\theta)^i, \tag{17}$$

where, $J^{ik} \triangleq \{j : y_j^{ik} = 1\}$ and $L^{ik} \triangleq \{j : y_j^{ik} = 0\}$, the operator $|\cdot|$ corresponds to cardinality and K^i is the number of integer solutions analysed in a specific critical region.

Equations 16 and 17 exclude integer solutions already visited, and integer solutions with higher values than the current upper bound, z_S, respectively.

For (ii), since the optimal value functions for the slave problem are linear, the comparison procedure described in Acevedo and Pistikopoulos [1] is used.

The algorithm terminates when the master problem, (6), is declared infeasible. The algorithmic steps are summarised in Fig. 1 and Table 4.

Next, we apply the steps of the proposed algorithm to an illustrative example.

2.4 Example 1

In a chemical engineering company, the decision maker has to choose between Reactor I, which is expensive but has a high rate of conversion, and Reactor II, which is more economic but has a lower rate of conversion, Fig. 2 (adapted from [13]).

Due to the presence of uncertainty in the cost coefficients, the multi-parametric OFC MILP problem results:

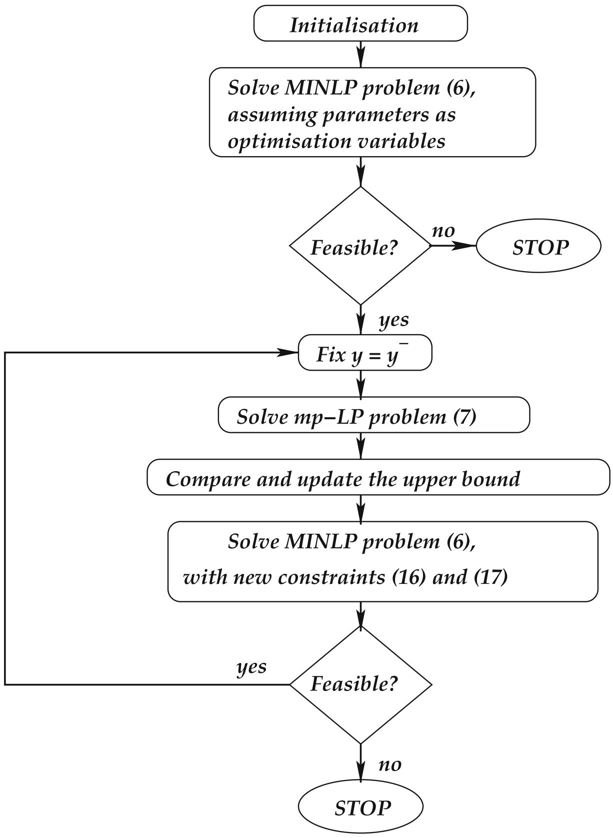


Fig. 1 Algorithm for OFC mp-MILP problems

Table 4 Steps of the algorithm for OFC mp-MILP problems

Step 0.	(Initialization) Define an initial region of Θ, CR, with best upper bound $\hat{z}^*(\theta) = \infty$, and an initial integer solution, $\bar{y}$
Step 1.	(Slave subproblem—multiparametric LP problem) For each region with a new integer solution, $\bar{y}$: (a) solve the mp-LP subproblem (7) to obtain a set of parametric upper bounds, $\hat{z}(\theta) = z_S^*$, and the corresponding critical regions *CR*; (b) if $\hat{z}(\theta) \leq z_S^*(\theta)$ for some region of θ, update the best upper bound function, $\hat{z}(\theta)$, and the corresponding integer solutions, y^*; (c) if an infeasibility is found in some region *CR*, go to *Step 2*
Step 2.	(Master subproblem—MINLP problem) For each region *CR*, formulate and solve to global optimality the MINLP master subproblem, (6), (i) treating θ as an optimisation variable, (ii) introducing an integer cut (16) and (iii) introducing a parametric cut (17). Return to *Step 1* with new integer solutions and corresponding *CR*s
Step 3.	(Convergence) The algorithm terminates in a region where the solution of the master MINLP subproblem is infeasible. Then, the optimal parametric solution is given by the current upper bounds $\hat{z}^*(\theta)$

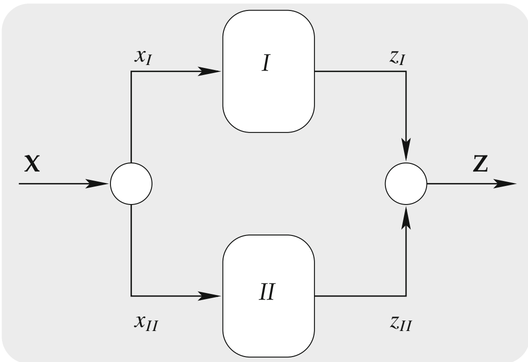


Fig. 2 Superstructure of illustrative example 1

$$\begin{aligned}
\min_{x_1, x_2, y_I, y_{II}} \quad & (6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 \\
& + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II}, \\
s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 = 10, \\
& x_1 \leq 20y_I, \\
& x_2 \leq 20y_{II}, \\
& x_1, x_2 \geq 0, \\
& 0 \leq \theta_1, \ \theta_2 \leq 20, \\
& y_I, y_{II}, \in \{0, 1\}.
\end{aligned} \tag{18}$$

The solution steps of the algorithm in Table 4 and Fig. 1 are listed next.

Step 0. (initialization) Solve Problem (18) considering θ as being optimisation variables, $\bar{y} = (1, 0)$.

Step 1. ($k = 1$, slave subproblem) Fix $y = \bar{y}$. The mp-LP problem in (7) is formulated as:

$$\begin{aligned} \min_{x_1, x_2} \quad & (6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 + (7.5 + 0.3\theta_1), \\ s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 = 10, \\ & 0 \le x_1 \le 20, 0 \le x_2 \le 0, \\ & 0 \le \theta_1, \theta_2 \le 20, \\ & y_I, y_{II}, \in \{0, 1\}. \end{aligned} \tag{19}$$

The solution of (19) is computed using the OFC mp-LP algorithm:

$$\begin{cases} x_1 = 12.5, \\ x_2 = 0, \\ 0 \le \theta_1 \le 20, \\ 0 \le \theta_2 \le 20. \end{cases}$$

Step 2. ($k = 1$, master subproblem) Solve the master problem in (6) with two additional constraints, due to (16) and (17):

$$y_1 - y_2 \le 0, \tag{20}$$

$$\begin{aligned} &(6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 \\ &\quad + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II} \le 3.425\theta_1 + 87.5. \end{aligned} \tag{21}$$

The solution is obtained using the commercial package GAMS/BARON [53]: $\bar{y} = (0, 1)$.

Step 1. ($k = 2$, slave subproblem) By fixing $y = (0, 1)$, the solution of (7) results in:

$$\begin{cases} x_1 = 0, \\ x_2 = 14.9254, \\ 0 \le \theta_1 \le 20, \\ 0 \le \theta_2 \le 20. \end{cases}$$

Step 1. ($k = 2$, comparison of solutions) Solutions valid in $0 \le \theta_1, \theta_2 \le 20$:

Solution 1	Solution 2
$x_1 = 12.5$	$x_1 = 0$
$x_2 = 0$	$x_2 = 14.9254$
$y_1 = 1$	$y_1 = 0$
$y_2 = 0$	$y_2 = 1$
$z = 87.5 + 3.4250\theta_1$	$z = 95.0524 + 2.6873\theta_2$

The intersection of the two planes is given by the line:

$$3.4250\theta_1 - 2.6873\theta_2 = 7.524,$$

below which $z_S^1(\theta)$ is valid.

Table 5 Map of optimal parametric solutions for Example 1

Region	Solution
$\left\{\begin{array}{l}\theta_1 \geq 0\\ 0 \leq \theta_2 \leq 20\\ 3.4250\theta_1 - 2.6873\theta_2 \leq 7.524\end{array}\right\}_{CR1}$	$x_1 = 0$, $x_2 = 14.9254$, $y_1 = 0$, $y_2 = 1$
$\left\{\begin{array}{l}\theta_1 \leq 20\\ 0 \leq \theta_2 \leq 20\\ 3.4250\theta_1 - 2.6873\theta_2 \geq 7.524\end{array}\right\}_{CR2}$	$x_1 = 12.5 + 1.25\theta_2$, $x_2 = 0$, $y_1 = 1$, $y_2 = 0$

Step 2. ($k = 2$, master problem) Solve Problem (6) with four additional constraints:

$$y_1 - y_2 \leq 0, \tag{22}$$
$$y_2 - y_1 \leq 0, \tag{23}$$
$$(6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II} \leq 3.4250\theta_1 + 87.5, \tag{24}$$
$$(6.4 + 0.25\theta_1)x_1 + (6.0 + 0.17\theta_2)x_2 + (7.5 + 0.3\theta_1)y_I + (5.5 + 0.15\theta_2)y_{II} \leq 95.0524{+}2.6873\theta_2. \tag{25}$$

The resulting problem is infeasible, and hence, the algorithm terminates. The final solution is listed in Table 5.

3 Multi-parametric RIM MILP problems

In this section, we consider the formulation in (4), i.e. the general case with independent varying parameters both in the objective function and the right-hand side of the constraints, rewritten in a more compact mathematical form [34]:

$$\begin{aligned} z(\theta) = \min_{x,y} \quad & (c + H\theta)^T x + (d + L\theta)^T y, \\ s.t. \quad & Ax + Ey \leq b + F\theta, \\ & \Gamma x + \Phi y = \gamma + \Psi\theta, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^q, \\ & \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}, \end{aligned} \tag{26}$$

where, b, γ, F and Ψ are real matrices with appropriate dimensions. For the solution of (26), we present in the following an extension of the algorithm presented in Sect. 2, which iterates between two optimisation subproblems, a master MINLP problem and a slave multi-parametric problem. The principal difference is the comparison procedure between two parametric solutions, since in this case the optimal value function is non-linear.

3.1 Master problem

By considering the parameters θ as optimisation variables, (26) results in the following MINLP formulation [34]:

$$\begin{aligned} z_M(\theta) = \min_{x,\theta,y} \quad & (c + H\theta)^T x + (d + L\theta)^T y, \\ s.t. \quad & Ax + Ey \leq b + F\theta, \\ & \Gamma x + \Phi y = \gamma + \Psi\theta, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^q, \\ & \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}. \end{aligned} \tag{27}$$

Note that (27) involves bilinear terms in the objective function, thereby it is a non-convex problem which requires a global optimisation procedure. The solution of (27) returns a new binary vector, $y = \bar{y}$, to the slave problem, which is described next.

3.2 Slave problem

By fixing $y = \bar{y}$, the slave problem is formulated in the following way:

$$\begin{aligned} z_S(\theta) = (d + L\theta)^T \bar{y} + \min_{x} \quad & c^T x + \theta^T H^T x, \\ s.t. \quad & Ax \leq b' + F\theta, \\ & \Gamma x \leq \gamma' + \Psi\theta, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^m, \\ & \theta \in \Theta = \{\theta : \theta \in \mathbb{R}^s, G\theta \leq e\}. \end{aligned} \tag{28}$$

where, $b' = (b - E\bar{y})$ and $\gamma' = (\gamma - \Phi\bar{y})$. Once again, Problem (28) is solved using a modified version of the original mp-LP algorithm [20]. As shown before, applying Eq. 11 to problem (28) results in:

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} - \begin{bmatrix} 0 & A^T & \Gamma^T \\ \Lambda A & diag\,(g) & 0 \\ \Gamma & 0 & 0 \end{bmatrix}_{\eta_0}^{-1} \cdot \begin{bmatrix} H^T \\ F \\ \Psi \end{bmatrix}_{\eta_0} \cdot (\theta - \theta_0). \tag{29}$$

The RIM mp-LP algorithm was implemented in Matlab.

Remark 2 Note that in Eq. 29, contrary to (12), N_0 is a full rank matrix and therefore the explicit expression of the optimisation variables depends on the parameters.

3.3 The algorithm

Between every master–slave iteration, we need to (i) introduce integer and parametric cuts in the master MINLP problem (Eqs. 16 and 17, respectively), and (ii) compare the parametric solutions obtained in the slave problem in order to retain the best. While the cuts are identical to the OFC problem, the comparison of different solutions of the slave problem is itself a global optimisation problem, since in this case the optimal value functions are non-linear. Here, we address this issue by storing all different optimal solutions valid inside overlapping regions and computing the optimum solution online by direct value comparisons (enclosure of all solutions—see [20]).

The algorithm terminates when the master problem is infeasible.

Table 6 Steps of the algorithm for RIM mp-MILP problems

Step 0.	(Initialization) Define an initial region of Θ, CR, with best upper bound $\hat{z}^*(\theta) = \infty$, and an initial integer solution, $\bar{y}$
Step 1.	(Slave subproblem—multiparametric LP problem) For each region with a new integer solution, $\bar{y}$: (a) solve the mp-LP subproblem (28) to obtain a set of parametric upper bounds, $\hat{z}(\theta) = z_S^*$, and the corresponding critical regions *CR*; (b) if $\hat{z}(\theta) \leq z_S^*(\theta)$ for some region of θ, update the best upper bound function, $\hat{z}(\theta)$, and the corresponding integer solutions, y^*; (c) if an infeasibility is found in some region *CR*, go to *Step 2*
Step 2.	(Master subproblem—MINLP problem) For each region *CR*, formulate and solve to global optimality the MINLP master subproblem, (27), (i) treating θ as an optimisation variable, (ii) introducing an integer cut (16) and (iii) introducing a parametric cut (17). Return to *Step 1* with new integer solutions and corresponding *CR*s
Step 3.	(Convergence) The algorithm terminates in a region where the solution of the master MINLP subproblem is infeasible. Then, the optimal parametric solution is given by the current upper bounds $\hat{z}^*(\theta)$

The algorithmic steps are summarised in Table 6, and are described in detailed in two illustrative problems, shown next.

3.4 Example 2

Consider again Example 1 in Fig. 2, but now with uncertainty involving both the customer's demand and the objective function, as follows:

$$
\begin{aligned}
\min_{x_1,x_2,y_I,y_{II}} \quad & (6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I + 5.5y_{II}, \\
s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 \geq 10 + \theta_2, \\
& x_1 \leq 40y_I, \\
& x_2 \leq 40y_{II}, \\
& x_1, x_2 \geq 0, \\
& 0 \leq \theta_1 \leq 20, \\
& 0 \leq \theta_2 \leq 10, \\
& y_I, y_{II}, \in \{0, 1\}.
\end{aligned}
\tag{30}
$$

We apply the solution steps of the proposed algorithm in Table 6.

Step 0. (initialisation) Solve Problem (30) considering θ as being optimisation variables, $\bar{y} = (1, 0)$.

Step 1. ($k = 1$, slave subproblem) Fix $y = \bar{y}$. The RIM mp-LP problem is formulated as:

$$
\begin{aligned}
\min_{x_1,x_2} \quad & (6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1), \\
s.t. \quad & 0.8 \cdot x_1 + 0.67 \cdot x_2 \geq 10 + \theta_2, \\
& 0 \leq x_1 \leq 40, \quad 0 \leq x_2 \leq 0, \\
& 0 \leq \theta_1 \leq 20, \quad 0 \leq \theta_2 \leq 10.
\end{aligned}
\tag{31}
$$

The solution of (31) is computed using the RIM mp-LP algorithm:

$$\begin{cases} x_1 = 12.5 + 1.25 \cdot \theta_2 \\ x_2 = 0, \\ 0 \le \theta_1 \le 20, \\ 0 \le \theta_2 \le 10. \end{cases}$$

Step 2. ($k = 1$, master subproblem) Solve the master problem in (27) with two additional constraints, due to (16) and (17):

$$y_1 - y_2 \le 0, \tag{32}$$

$$(6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I + 5.5y_{II} \le 87.5 + 3.425\theta_1 - 0.3125\theta_1\theta_2 + 8\theta_2. \tag{33}$$

The solution is obtained using the commercial package GAMS/BARON [53]: $\bar{y} = (0, 1)$.

Step 1. ($k = 2$, slave problem) By fixing $y = \bar{y}$, the solution of (28) is:

$$\begin{cases} x_1 = 0, \\ x_2 = 14.9254 + 1.4925\theta_2, \\ 0 \le \theta_1 \le 20, 0 \le \theta_2 \le 10. \end{cases}$$

Step 1. ($k = 2$, comparison of solutions) Solutions valid in $0 \le \theta_1 \le 20, 0 \le \theta_2 \le 10$:

Solution 1	Solution 2
$x_1 = 12.5 + 1.25\theta_2$	$x_1 = 0$
$x_2 = 0$	$x_2 = 14.9254 + 1.49254\theta_2$
$y_1 = 1$	$y_1 = 0$
$y_2 = 0$	$y_2 = 1$
$z = 87.5 + 3.425\theta_1 - 0.3125\theta_1\theta_2 + 8\theta_2$	$z = 8.955\theta_2 + 95.0524$

In this specific case, we can compute the intersection of the two solutions:

$$-0.955\theta_2 - 0.3125\theta_1\theta_2 - 7.5524 + 3.425\theta_1 = 0.$$

Otherwise, we store all parametric solutions of the slave problems and compute on-line the best decision.

Step 2. ($k = 2$, master problem) Solve Problem (27) with four additional constraints:

$$y_1 - y_2 \le 0, \tag{34}$$

$$y_2 - y_1 \le 0, \tag{35}$$

$$(6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I + 5.5y_{II} \le 87.5 + 3.425\theta_1 - 0.3125\theta_1\theta_2 + 8\theta_2, \tag{36}$$

$$(6.4 + 0.25\theta_1)x_1 + 6.0x_2 + (7.5 + 0.3\theta_1)y_I + 5.5y_{II} \le 8.955\theta_2 + 95.0524. \tag{37}$$

The resulting problem is infeasible, and thus, the algorithm terminates. The final solution is listed in Table 7.

Table 7 Map of critical regions for Problem (30)

Region	Solution
$\left\{\begin{array}{l} 0 \le \theta_1 \le 20 \\ 0 \le \theta_2 \le 10 \\ -0.955\theta_2 - 0.3125\theta_1\theta_2 + 3.425\theta_1 \le 7.5524 \end{array}\right\}_{CR1}$	$x_1 = 12.5 + 1.25\theta_2$ $x_2 = 0$ $y_1 = 1$ $y_2 = 0$
$\left\{\begin{array}{l} 0 \le \theta_1 \le 20 \\ 0 \le \theta_2 \le 10 \\ 0.955\theta_2 + 0.3125\theta_1\theta_2 - 3.425\theta_1 \le -7.5524 \end{array}\right\}_{CR2}$	$x_1 = 0$ $x_2 = 14.9254 + 1.4925\theta_2$ $y_1 = 0$ $y_2 = 1$

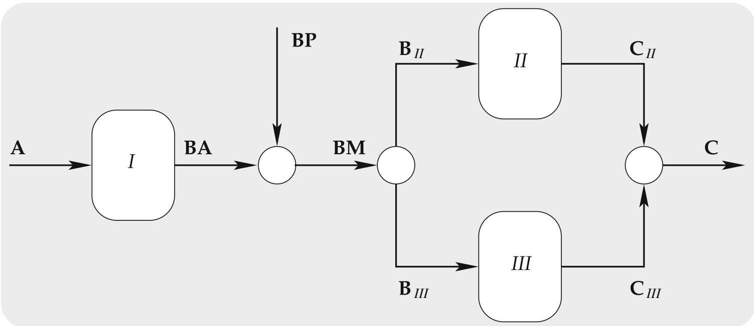


Fig. 3 Superstructure of the illustrative example 3

3.5 Example 3

This example is a variant of a process synthesis problem described by Biegler et al. [13], shown in Fig. 3. A chemical product C is produced using either process unit II or III, both of which use chemical B as raw material; on the other hand, B can either be directly purchased or manufactured using process I and purchasing raw material A.

Moreover, the decision is subject to uncertainty in the operation cost and product demand. The uncertainty—θ_1, θ_2—is assumed to be unstructured and bounded. The multi-parametric RIM MILP problem is posed as follows:

$$\begin{aligned} \min\ & -18 \cdot C + (10 \cdot y_I + 15 \cdot y_{II} + 20 \cdot y_{III}) \\ & +(2.5 \cdot A \cdot y_I + (4 + \theta_1) \cdot B_{II} \cdot y_{II} + 5.5 \cdot B_{III} \cdot y_{III}), \\ s.t.\ & C = 0.82 \cdot B_{II} + 0.95 \cdot B_{III}, \\ & 2 \le C \le 5 + \theta_2, \\ & A \le 16 \cdot y_I, \\ & y_{II} + yIII \ge 1, \\ & B_{II} - 30 \cdot y_{II} \le 0, \\ & B_{III} - 30 \cdot y_{III} \le 0, \end{aligned}$$

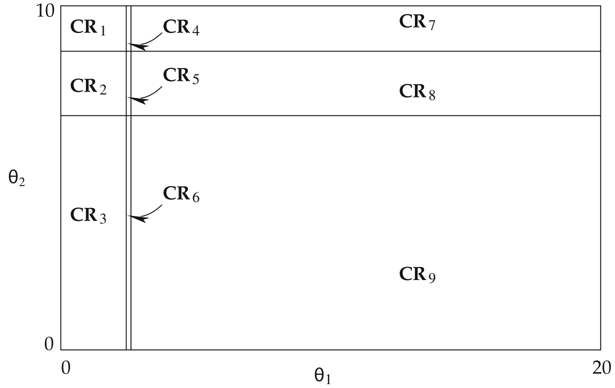


Fig. 4 Map of critical regions of Problem (38)

$$\begin{aligned} & B_{II} + B_{III} - BP - 0.9 \cdot A = 0, \\ & 0 \le \theta_1 \le 5, \\ & 0 \le \theta_2 \le 10, \\ & y_I, y_{II}, y_{III} \in \{0, 1\}, \\ & C, A, BP, B_{II}, B_{III} \ge 0. \end{aligned} \tag{38}$$

The final solution is depicted in Fig. 4 and Table 8.

Remark 3 Although we focus on OFC and RIM classes of mp-MILP problems, the procedure is still valid when matrices E, Φ also depend linearly on the parameters, as follows:

$$\begin{aligned} z(\theta) = \min_{x,y} \quad & (c + H\theta)^T x + (d + L\theta)^T y, \\ s.t. \quad & Ax + (e_1 + E_2\theta)y \le b + F\theta, \\ & \Gamma x + (\phi_1 + \Phi_2\theta)y = \gamma + \Psi\theta, \\ & x \in X \subseteq \mathbb{R}^n, \quad y \in \{0, 1\}^q, \\ & \theta \in \Theta, \end{aligned} \tag{39}$$

because, fixing the binary vector, $y = \bar{y}$, to the solution obtained in the master subproblem, (39) is rewritten as a RIM mp-LP problem:

$$\begin{aligned} z_S(\theta) = (d + L\theta)^T \bar{y} + \min_{x} \quad & c^T x + \theta^T H^T x, \\ s.t. \quad & Ax \le b' + F'\theta, \\ & \Gamma x \le \gamma' + \Psi'\theta, \\ & x \in X \subseteq \mathbb{R}^n, \ y \in \{0, 1\}^m, \\ & \theta \in \Theta, \end{aligned} \tag{40}$$

where, $b' = (b - e_1\bar{y})$, $\gamma' = (\gamma - \phi_1\bar{y})$, $F' = (F - E_2\bar{y})$ and $\Psi' = (\Psi - \Phi_2\bar{y})$.

4 Concluding remarks

We have presented a novel optimisation framework for the global solution of general mp-MILP problems, involving uncertainty in the objective function and the right-hand side of

Table 8 Solution of Problem 38 ($x = \Omega\theta + \omega$)

	Region	Integer solution $\bar{y}$	Continuous solution Ω	ω
⋮				
CR7	$2.42 \leq \theta_1 \leq 5$ $8.68 \leq \theta_2 \leq 10$	$\{1, 0, 1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1.0526 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 16 \\ -9.1368 \\ 0 \\ 5.2632 \end{bmatrix}$
CR8	$2.42 \leq \theta_1 \leq 5$ $6.81 \leq \theta_2 \leq 8.68$	$\{1, 0, 1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 1.1696 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 5.8480 \\ 0 \\ 0 \\ 5.2632 \end{bmatrix}$
CR9	$2.42 \leq \theta_1 \leq 5$ $0 \leq \theta_2 \leq 6.80$	$\{1, 0, 1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1.0526 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 16 \\ -9.1368 \\ 0 \\ 5.2632 \end{bmatrix}$
		$\{0, 0, 1\}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1.0526 \\ 0 & 0 \\ 0 & 1.0526 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 0 \\ 5.2632 \\ 0 \\ 5.2632 \end{bmatrix}$

the constraints. Based on our previous work on multi-parametric programming [18,21,46], a novel mp-LP algorithm was developed, which overcomes the presence of the non-convexities due to bilinear terms. This is then used in an efficient procedure, which iterates between a master MINLP subproblem, solved to global optimality, and a slave mp-LP subproblem. A number of examples are also presented.

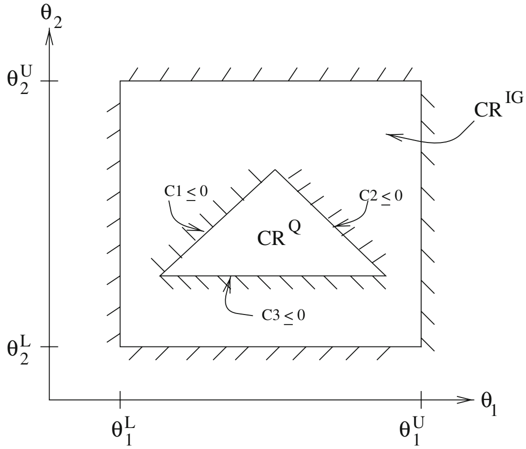
The proposed approach has many applications in hybrid and robust control—a topic which is currently being investigated and will be reported elsewhere [35].

Acknowledgements Financial support from EPSRC (GR/T02560/01) and Marie Curie European Project PRISM (MRTN-CT-2004-512233) is gratefully acknowledged.

Appendix: Definition of rest of the region

Given an initial region, CR^{IG} and a region of optimality, CR^{Q} such that $CR^{Q} \subseteq CR^{IG}$, a procedure is described in this section to define the rest of the region, $CR^{rest} = CR^{IG} - CR^{Q}$. For the sake of simplifying the explanation of the procedure, consider the case when only two parameters, θ_1 and θ_2, are present (see Fig. 5), where CR^{IG} is defined by the inequalities: $\{\theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2 \leq \theta_2^U\}$ and CR^{Q} is defined by the inequalities: $\{C1 \leq 0, C2 \leq 0, C3 \leq 0\}$ where $C1$, $C2$ and $C3$ are linear in θ. The procedure consists of considering one by one the inequalities which define CR^{Q}. Considering, for example, the inequality $C1 \leq 0$, the rest of the region is given by, CR_1^{rest} : $\{C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U\}$, which is obtained by reversing the sign of inequality $C1 \leq 0$ and removing redundant constraints in CR^{IG} (see Fig. 6). Thus, by considering the rest of the inequalities, the complete rest of the region is given by: $CR^{rest} = \{CR_1^{rest} \cup CR_2^{rest} \cup CR_3^{rest}\}$, where CR_1^{rest}, CR_2^{rest} and CR_3^{rest} are

Fig. 5 Critical regions, CR^{IG} and CR^{Q}



given in Table 9 and are graphically depicted in Fig. 7. Note that for the case when CR^{IG} is unbounded, simply suppress the inequalities involving CR^{IG} in Table 9.

Fig. 6 Division of critical regions—Step 1

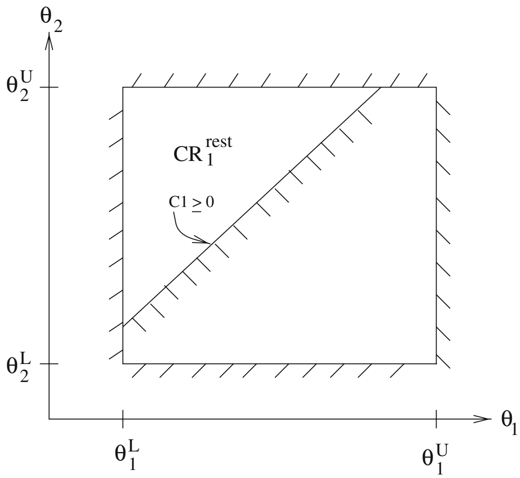


Table 9 Definition of rest of the regions

Region	Inequalities
CR_1^{rest}	$C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U$
CR_2^{rest}	$C1 \leq 0, C2 \geq 0, \theta_1 \leq \theta_1^U, \theta_2 \leq \theta_2^U$
CR_3^{rest}	$C1 \leq 0, C2 \leq 0, C3 \geq 0, \theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2$

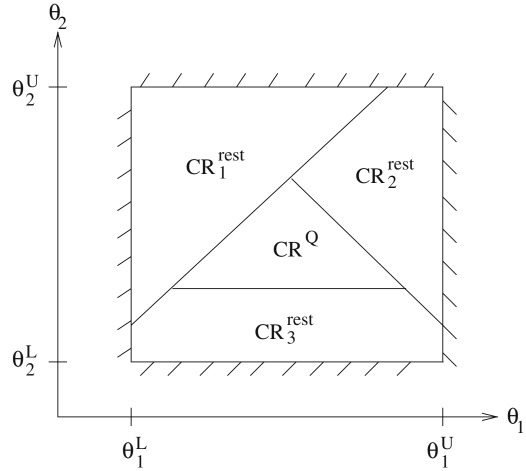


Fig. 7 Division of critical regions—rest of the regions

References

1. Acevedo, J., Pistikopoulos, E.N.: A multiparametric programming approach for linear process engineering problems under uncertainty. Ind. Eng. Chem. Res. **36**, 717–728 (1997)
2. Acevedo, J., Salgueiro, M.: An efficient algorithm for convex multiparametric nonlinear programming problems. Ind. Eng. Chem. Res. **42**(23), 5883–5890 (2003)
3. Adjiman, C.S., Androulakis, I.P., Floudas, C.A.: A global optimization method, αbb, for general twice-differentiable constrained nlps. ii. implementation and computational results. Comput. Chem. Eng. **22**(9), 1159–1179 (1998)
4. Adjiman, C.S., Dallwig, S., Floudas, C.A., Neumaier, A.: A global optimization method, αbb, for general twice-differentiable constrained nlps. i. theoretical advances. Comput. Chem. Eng. **22**(9), 1137–1158 (1998)
5. Armacost, R.L.: Computational experience in sensitivity analysis for nonlinear programming. Math. Program. **6**, 301–326 (1974)
6. Armacost, R.L.: Sensitivity analysis in parametric nonlinear programming. PhD thesis, School of Engineering and Applied Science, George Washington University, Washington, USA (1976)
7. Arnold, V.I.: Catastrophe Theory. Springer-Verlag, Berlin (1984)
8. Baotić, M., Christophersen, F.J., Morari, M.: Constrained optimal control of hybrid systems with linear performance index. IEEE Trans. Autom. Control **51**(12), 1903–1919 (2006)
9. Bazaraa, M.S., Shetty, C.M.: Nonlinear Programming—Theory and Algorithms. Wiley, New York (1979)
10. Bemporad, A., Filippi, C.: Suboptimal explicit receding horizon control via approximate multiparametric quadratic programming. J. Optim. Theory Appl. **117**(1), 9–38 (2003)
11. Bemporad, A., Morari, M., Dua, V., Pistikopoulos, E.N.: The explicit linear quadratic regulator for constrained systems. Automatica **38**, 3–20 (2002)
12. Bemporad, A., Borrelli, F., Morari, M.: Min–max control of constrained uncertain discrete-time linear systems. IEEE Trans. Autom. Control **48**(9), 1600–1606 (2003)
13. Biegler, L.T., Grossman, I.E., Westerberg, A.W.: Systematic Methods of Chemical Process Design. Prentice Hall, New Jersey (1997)
14. Borrelli, F., Bemporad, A., Morari, M.: Geometric algorithm for multiparametric linear programming. J. Optim. Theory Appl. **118**(3), 515–540 (2003)
15. Borrelli, F., Baotić, M., Bemporad, A., Morari, M.: Dynamic programming for constrained optimal control of discrete-time linear hybrid systems. Automatica **41**, 1709–1721 (2005)
16. Dua, V.: Parametric programming techniques for process engineering problems under uncertainty. PhD thesis. Department of Chemical Engineering and Chemical Technology, Imperial College of Science, Technology and Medicine London, London, UK (2000)
17. Dua, V., Pistikopoulos, E.N.: Algorithms for the solution of multiparametric mixed-integer nonlinear optimization problems. Ind. Eng. Chem. Res **38**, 3976–3987 (1999)
18. Dua, V., Pistikopoulos, E.N.: An algorithm for the solution of multiparametric mixed integer linear programming problems. Ann. Oper. Res. **99**, 123–139 (2000)

19. Dua, P., Pistikopoulos, E.N.: Modelling and control of drug delivery systems. Comput. Chem. Eng. **29**(11–12), 2290–2296 (2005)
20. Dua, V., Bozinis, N.A., Pistikopoulos, E.N.: A multiparametric programming approach for mixed-integer quadratic engineering problems. Comput. Chem. Eng. **26**, 715–733 (2002)
21. Dua, V., Papalexandri, K.P., Pistikopoulos, E.N.: Global optimization issues in multiparametric continuous and mixed-integer optimization problems. J. Glob. Optim. **30**, 59–89 (2004)
22. Dua, P., Doyle, F.J., Pistikopoulos, E.N.: Model-based blood glucose control for type 1 diabetes via parametric programming. IEEE Trans. Biomed. Eng. **53**(8), 1478–1491 (2006)
23. Faísca, N.P., Saraiva, P.M., Rustem, B., Pistikopoulos, E.N.: A multi-parametric programming approach for multi-level hierarchical and decentralised optimisation problems. Comput. Manag. Sci. (published electronically) (2007)
24. Faísca, N.P., Dua, V., Saraiva, P.M., Rustem, B., Pistikopoulos, E.N.: Parametric global optimisation for bilevel programming. J. Glob. Optim. **38**(4), 609–623 (2007)
25. Faísca, N.P., Kouramas, K.I., Saraiva, P.M., Rustem, B., Pistikopoulos, E.N.: A multi-parametric programming approach for constrained dynamic programming problems. Optim. Lett. **2**(2), 267–280 (2008)
26. Fiacco, A.V.: Sensitivity analysis for nonlinear programming using penalty methods. Math. Program. **10**, 287–311 (1976)
27. Fiacco, A.V.: Introduction to Sensitivity and Stability Analysis in Nonlinear Programming. Academic Press, New York (1983)
28. Fiacco, A.V., Kyparisis, J.: Computable bounds on parametric solutions of convex problems. Math. Program. **40**, 213–221 (1988)
29. Filippi, C.: An algorithm for approximate multiparametric linear programming. J. Optim. Theory Appl. **120**(1), 73–95 (2004)
30. Floudas, C.A.: Deterministic Global Optimization. Kluwer, Dordrecht (2000)
31. Gal, T.: Rim multiparametric linear programming. Manag. Sci. **21**(5), 567–575 (1975)
32. Gal, T., Nedoma, J.: Multiparametric linear programming. Math. Program. Stud. **18**, 406–422 (1972)
33. Gass, S.I., Saaty, T.L.: Parametric objective function (part 2). J. Oper. Res. Soc. Am. **3**(4), 395–401 (1954)
34. Kosmidis, V.S.: Multiparametric analysis of mixed integer linear programming problems. Master's thesis, Imperial College London, University of London, London, UK (1999)
35. Kouramas, K.I., Faísca, N.P., Rustem, B., Pistikopoulos, E.N.: Design of robust multi-parametric controllers for constrained multi-stage optimization problems. Automatica (submitted)(2008)
36. Kyparisis, J., Fiacco, A.V.: Generalized convexity and concavity of the optimal value function in nonlinear programming. Math. Program. **39**, 285–304 (1987)
37. Li, Z., Ierapetritou, M.G.: A new methodology for the general mixed-integer linear programming (milp) problems. Ind. Eng. Chem. Res. **46**, 5141–5151 (2007)
38. Li, Z., Ierapetritou, M.G.: Process scheduling under uncertainty using multiparametric programming. AIChE J **53**(12), 3183–3203 (2007)
39. Lundberg, B.N., Poore, A.B.: Numerical continuation and singularity detection methods for parametric nonlinear programming. SIAM J. Optim. **3**(1), 134–154 (1993)
40. Mangasarian, O.L., Fromovitz, S.: The Fritz John necessary optimality conditions in the presence of equality and inequality constraints. J. Math. Anal. Appl. **17**, 37–47 (1967)
41. McCormick, G.P.: Optimality criteria in nonlinear programming. In: SIAM-AMS Proceedings 1975, vol. 9, p. 27 (1976)
42. Morari, M., Barić, M.: Rcent developments in the control of constrained hybrid systems. Comput. Chem. Eng. **30**(10–12), 1619–1631 (2006)
43. Pertsinidis, A.: On the parametric optimization of mathematical programs with binary variables and its application in the chemical engineering process synthesis. PhD thesis, Department of Chemical Engineering, Carnegie-Mellon University, Pittsburg, USA (1992)
44. Pertsinidis, A., Grossman, I.E., McRae, G.J.: Parametric optimization of milp programs and a framework for the parametric optimization of minlps. Comput. Chem. Eng. **22**, S205–S212 (1998)
45. Pistikopoulos, E.N., Dua, V., Bozinis, N.A., Bemporad, A., Morari, M.: On-line optimization via off-line parametric optimization tools. Comput. Chem. Eng. **24**, 183–188 (2000)
46. Pistikopoulos, E.N., Georgiadis, M.C., Dua, V.: Multi-Parametric Programming: Theory, Algorithms, and Applications, vol. 1. Wiley-VCH, Weinheim (2007)
47. Pistikopoulos, E.N., Georgiadis, M.C., Dua, V.: Multi-Parametric Model-Based Control: Theory and Applications, vol. 2. Wiley-VCH, Weinheim (2007)
48. Pistikopoulos, E.N., Faísca, N.P., Georgiadis, M.C.: Advanced optimization techniques in the supply chain management of manufacturing industries. In: PulPaper 2007. Finish Paper Engineers' Association, Helsinki, Finland (2007)

49. Poore, A.B., Tiahrt, C.A.: Bifurcation problems in nonlinear parametric programming. Math. Program. **39**, 189–205 (1987)
50. Ryu, J.H., Pistikopoulos, E.N.: A novel approach to scheduling of zero-wait batch processes under processing time variations. Comput. Chem. Eng. **31**(3), 101–106 (2007)
51. Ryu, J.H., Dua, V., Pistikopoulos, E.N.: A bilevel programming framework for enterprise-wide process networks under uncertainty. Comput. Chem. Eng. **28**, 1121–1129 (2004)
52. Saaty, T.L., Gass, S.I.: Parametric objective function (part 1). J. Oper. Res. Soc. Am. **2**(3), 316–319 (1954)
53. Sahinidis, N.V.: BARON: User's Manual Version 4.0. Department of Chemical Engineering, University of Illinois, Urbana-Campaign (2000)
54. Sakizlis, V., Dua, V., Perkins, J.D., Pistikopoulos E.N.: The explicit control law for hybrid systems via parametric programming. In: Proceedings of the American Control Conference, Anchorage, AK, pp. 674–679 (2002)
55. Sakizlis, V., Perkins, J.D., Pistikopoulos, E.N.: Parametric controllers in simultaneous process and control design optimization. Ind. Eng. Chem. Res. **42**, 4545–4563 (2003)
56. Sakizlis, V., Perkins, J.D., Pistikopoulos, E.N.: Recent advances in optimization-based simulataneous process and control design. Comput. Chem. Eng. **28**, 2069–2086 (2004)
57. Sakizlis, V., Kakalis, M.P., Dua, V., Perkins, J.D., Pistikopoulos, E.N.: Design of robust controllers via parametric programming. Automatica **40**, 189–201 (2004)
58. Sakizlis, V., Kakalis, N.M.P., Dua, V., Perkins, J.D., Pistikopoulos, E.N.: Design of robust model-based controllers via parametric programming. Automatica **40**(2), 189–201 (2004)
59. Sakizlis, V., Perkins, J.D., Pistikopoulos, E.N.: Explicit solutions to optimal control problems for constrained continuous-time linear systems. IEE Proc. Control Theory Appl. **152**(4), 443–452 (2005)
60. Sakizlis, V., Kouramas, K.I., Faisca, N.P., Pistikopoulos, E.N.: Towards the design of parametric model predictive controllers for non-linear constrained systems. In: Assessment and Future Directions of Nonlinear Model Predictive Control. Lecture Notes in Control and Information Sciences, vol. 358. Springer, Berlin (2007)
61. Smith, E.M.B., Pantelides, C.C.: A symbolic reformulation: spatial branch-and-bound algorithm for the global optimisation of nonconvex minlps. Comput. Chem. Eng. **23**, 457–478 (1999)
62. Tøndel, P., Johansen, T.A., Bemporad, A.: An algorithm for multi-parameffic quadratic programming and explicit mpc solutions. Automatica **39**(3), 489–497 (2003)